Probability and Expected Value
Part 1: Probability States

Howard Halim

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Introduction

In many probability questions that show up on the AMC / AIME / HMMT / ARML / PUMaC / [insert any computation-based contest here], there is a process that repeats and cycles through several *states*, and you are asked you calculate the probability of getting from a starting state to a specific 'target' end state. A common technique for solving these problems is called *probability states*, where you calculate the probability of getting to the target state from any possible state.

Examples

1. AMC 12A 2017:
   A square is drawn in the coordinate plane with vertices at $(2,2)$, $(-2,2)$, $(-2,-2)$, $(2,-2)$. A particle starts at $(0,0)$. Every second, it moves to one of the eight lattice points closest to its current position, with equal probability. The particle will eventually hit the square for the first time, either at one of the 4 corners of the square or at at an edge. What is the probability that it will hit a corner before an edge?

2. Mr. Gregson is on a 10 m bridge, 3 m from the left side. Each minute, he moves 1 m left or right. He continues until he reaches one of the two ends of the bridge, at which point he leaves the bridge. What is the probability that he will leaves the bridge at the left end?

3. AHSME 1981:
   Alice, Bob, and Carol repeatedly take turns tossing a die. Alice begins; Bob always follows Alice; Carol always follows Bob; and Alice always follows Carol. Find the probability that Carol will be the first one to toss a six.

Expected Value With States

You can also use states to solve expected value problems. Instead of calculating the probability for each state, you calculate the expected value if you start from that state.

4. You roll a die until a 6 shows up. What is the expected number of rolls it will take?

5. HMMT 2015:
Consider an 8×8 grid of squares. A rook is placed in the lower left corner, and every minute it moves to a square in the same row or column with equal probability (the rook must move; i.e. it cannot stay in the same square). What is the expected number of minutes until the rook reaches the upper right corner?

6. An ant starts at one vertex of a cube, and every minute it moves along an edge to an adjacent vertex. What is the expected number of minutes before the ant arrives at the diagonally opposite vertex of the cube?

7. We have two bins, A and B. Initially there are 3 balls in bin A and no balls in bin B. We proceed with a series of moves as follows: on each move, one of the three balls is randomly chosen, and it is moved from its bin to the other bin. Find the expected number of moves until the first time that all of the balls are simultaneously in bin B.

Practice Problems

8. HMMT 2016:
Michael is playing basketball. He makes 10% of his shots, and gets the ball back after 90% of his missed shots. If he does not get the ball back he stops playing. What is the probability that Michael eventually makes a shot?

9. PUMaC 2017:
The four faces of a tetrahedral die are labelled 0, 1, 2, and 3, and the die has the property that, when it is rolled, the die promptly vanishes, and a number of copies of itself appear equal to the number on the face the die landed on. Suppose the die and all its copies are continually rolled. Find the probability that they will all eventually disappear.

10. Now, Mr. Wilson is on a 10 m bridge, 3 m from the left side, moving 1 m left or right each minute (same conditions as problem 2). However, Mr. Wilson refuses to get off the bridge at the right side! If he reaches the right side, on the next minute he will always move left. Find the expected number of minutes until he leaves the bridge (on the left side).

11. ARML Local 2015:
Johanna and Edward play a game with a 6-sided die. For each turn, a player rolls a die until they roll a 1, in which case they lose, or until they roll a 5 or 6, in which case their turn is over and the other player rolls the die under the same rules. The game continues until some player rolls a 1 and loses. If Johanna plays first, compute the probability that she wins the game.

12. HMMT 2009:
    Mario is once again on a quest to save Princess Peach. Mario enters Peachs castle and finds himself in a room with 4 doors. This room is the first in a sequence of 2 indistinguishable rooms. In each room, 1 door leads to the next room in the sequence (or, for the second room, into Bowsers level), while the other 3 doors lead to the first room.

    (a) Suppose that in every room, Mario randomly picks a door to walk through. What is the expected number of doors (not including Marios initial entrance to the first room) through which Mario will pass before he reaches Bowsers level?

    (b) Suppose that instead there are 6 rooms with 4 doors. In each room, 1 door leads to the next room in the sequence (or, for the last room, Bowsers level), while the other 3 doors lead to the first room. Now what is the expected number of doors through which Mario will pass before he reaches Bowsers level?

    (c) In general, if there are d doors in every room (but still only 1 correct door) and r rooms, the last of which leads into Bowsers level, what is the expected number of doors through which Mario will pass before he reaches Bowsers level?