Probability and Expected Value
Part 2: Linearity of Expectation

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Definitions

A *random variable* X is a variable that has multiple possible values $x_1, x_2, x_3, \ldots, x_n$ with probabilities $p_1, p_2, p_3, \ldots, p_n$, respectively. Some examples are the outcome of rolling a die, or flipping a coin.

The *expected value* of X, denoted $E[X]$, is the average value of X. It is calculated as the weighted average of $x_1, x_2, x_3, \ldots, x_n$, where the weights are the probabilities $p_1, p_2, p_3, \ldots, p_n$:

$$E[X] = p_1x_1 + p_2x_2 + p_3x_3 + \cdots p_nx_n = \sum_{i=1}^{n} p_ix_i.$$

Linearity of Expectation

Linearity of Expectation is a powerful theorem, which says that if a random variable X is the sum of two random variables Y and Z, then

$$E[X] = E[Y] + E[Z].$$

In general, if X is a linear combination of random variables, i.e.

$$X = a_1X_1 + a_2X_2 + a_3X_3 + \cdots a_nX_n,$$

then

$$E[X] = a_1E[X_1] + a_2E[X_2] + a_3E[X_3] + \cdots a_nE[X_n].$$

This is very useful when the expected value of X is difficult to compute directly, but the expected value of the X_i's is easy to find. In many contest problems about expected value, **finding good choices of the X_i's makes the problem much easier.**

Examples

1. Three coins are flipped. What is the expected number of heads that appear?

2. Mr. Ianine is standing on an infinitely long bridge. Each minute, he moves 1 m left with probability 1/3, and 1 m right with probability 2/3. What is his expected location after an hour?

3. Let X be a 5-digit number, whose digits are a random permutation of $1,2,3,4,5$. What is the expected value of X?

4. A random 5-digit number is chosen. What is the expected number of different digits? (For this question, numbers with leading zeros like 00585 are considered 5-digit numbers)

5. A coin is flipped 100 times. What is the expected number of consecutive pairs of heads? (If the sequence is HHTHHH, there are 3 pairs of consecutive heads)

6. If I shuffle a deck of cards randomly, what is the expected number of cards that will end up in their same position?

Practice Problems

7. 100 dice are rolled, and the 100 results are added up. What is the expected value of the sum?

8. Bob has a box of 4 balls - one green, one blue, one yellow, and one red. Each day of the week, he takes out a ball at random, records its color, and puts it back into the box. What is the expected number of distinct colors that he will take out of the box?

9. HMMT 2013:
   The digits $1,2,3,4,5,6$ are randomly chosen (without replacement) to form the three-digit numbers $M = \overline{ABC}$ and $N = \overline{DEF}$. For example, we could have $M = 413$ and $N = 256$. Find the expected value of $M \cdot N$.

10. HMMT 2006:
    At a nursery, 2006 babies sit in a circle. Suddenly, each baby randomly pokes either the baby to its left or to its right. What is the expected value of the number of unpoked babies?

11. AHSME 1989:
    Suppose that 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row $GBBGGGBGBGGGBGBGGBGG$ we have $S = 12$. Find the expected value of S.

Challenge Problems

12. AIME 2006:
Let $\mathcal{S}$ be the set of real numbers that can be represented as repeating decimals of the form $0.\overline{abc}$ where a, b, c are distinct digits. Find the sum of the elements of $\mathcal{S}$.

13. PUMaC 2017:
Charlies graduating class has 10 homerooms each of size 16, into which students are randomly assigned. At graduation, the homerooms are randomly numbered and the students in the first homeroom are called alphabetically, followed by the second homeroom, etc. If Charlie is 54th overall in her class alphabetically, find the expected number of students to get called before her.

14. NIMO 4.3:
One day, a bishop and a knight were on squares in the same row of an infinite chessboard, when a huge meteor storm occurred, placing a meteor in each square on the chessboard independently and randomly with probability p. Neither the bishop nor the knight were hit, but their movement may have been obstructed by the meteors. For what value of p is the expected number of valid squares that the bishop can move to (in one move) equal to the expected number of squares that the knight can move to (in one move)?

15. HMMT 2017:
New this year at HMNT: the exciting game of RNG baseball! In RNG baseball, a team of infinitely many people play on a square field, with a base at each vertex; in particular, one of the bases is called the home base. Every turn, a new player stands at home base and chooses a number n uniformly at random from $\{0, 1, 2, 3, 4\}$. Then, the following occurs:

    (a) If $n > 0$, then the player and everyone else currently on the field moves (counterclockwise) around the square by n bases. However, if in doing so a player returns to or moves past the home base, he/she leaves the field immediately and the team scores one point.

    (b) If $n = 0$ (a strikeout), then the game ends immediately; the team does not score any more points.

    What is the expected number of points that a given team will score in this game?

16. PUMaC 2016:
A knight is placed at the origin of the Cartesian plane. Each turn, the knight moves in an chess L-shape (2 units parallel to one axis and 1 unit parallel to the other) to one of eight possible locations, chosen at random. After 2016 such turns, what is the expected value of the square of the distance of the knight from the origin?