Invariants and Monovariants

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Invariants

Some math problems involve a process, where a certain step is repeated many (possibly infinite) times. Usually, a starting state is given, and each time this step is repeated, it changes to a new state. Sometimes the process stops at an end state. When solving these problems, *it is useful to find something that does not change* throughout this process. This is called an invariant.

Example 1

A circle is divided into 6 sectors. The numbers 1, 0, 1, 0, 0, and 0 are written in that order. You can increase two adjacent numbers by 1. Is it possible to make all numbers equal?

Example 2

Let n be an odd positive integer. The first $2n$ integers are written on the blackboard. Erase any two integers on the blackboard, and replace them with their difference. Eventually, there will be one number left. Prove that this number is odd.

Finding an invariant is especially useful if the question is asking whether a certain state can be reached or not, to show that the process must end in a particular state, or to find all possible end states.

Monovariants

Another concept related to the invariant is a monovariant. This is a quantity that only changes in one direction. When the step is repeated, it either always decreases or always increases. Usually it helps to find an upper or lower limit for the monovariant, to prove that the process must end. If you have to find an algorithm that gets to a specific end state, prove that a process will eventually end, or prove that it is impossible to return to the starting state, then you should try to find a monovariant.

Example 3

In the school UTS, each student has at most three enemies. Prove that the students can be split into two groups, such that each student has at most one enemy in his/her group.

Example 4

A real number is written in each cell of an $m \times n$ grid. You can flip the sign of all numbers in the same row or column. Prove that you make all rows and columns have a nonnegative sum.

Problems

1. Seven vertices of a cube are marked by 0 and one by 1. You may repeatedly select an edge and increase the numbers at the ends of that edge by 1. Can you make all numbers equal?

2. Take a regular 8×8 chessboard. Can you obtain only one black square if:

   (a) You can repaint all squares in a row or column.

   (b) You can repaint all squares in a 2×2 square.

3. Three numbers are written on a blackboard. We can choose one of them, say c, and replace it by $2a + 2b - c$, where a and b are the other two numbers. Can we reach the triple $20, 21, 24$ from the triple $1, 2, 3$?

4. Many handshakes are exchanged at a big international congress. Show that at any moment, there is an even number of people who have exchanged an odd number of handshakes.

5. Each of the numbers from 1 to 10^6 is repeatedly replaced by the sum of its digits, until we end up with a list of 10^6 one-digit numbers. Will there be more 1's or 2's in this list?

6. Is it possible to start with a knight at some corner of a chessboard and reach the opposite corner passing through all squares exactly once?

7. There is 1 stone at each vertex of a square. You can repeat this step as many times as you want: you can take away any number of stones from one vertex, and add twice as many stones to the piles at the adjacent vertices. Is it possible to get 2015, 2016, 2016, and 2017 stones at consecutive vertices?

8. Two opposite corners of an 8×8 chessboard are removed. Can the remaining figure be tiled with 31 dominoes? (Each domino is a 2×1 rectangle)

9. A bubble chamber contains three types of subatomic particles: 13 particles of type X, 15 of type Y, and 17 of type Z. When two different particles collide, they each turn into a particle of the third type. Can all the particles become the same type?

10. On an infinite (in both directions) number line, several stones are placed on integers (there can be multiple stones on the same number). You may remove two stones at n, and place one on each of $n - 1$ and $n + 1$. Is it possible to return to the starting configuration after a finite number of steps?

11. Remove the first digit of the number 7^{2016}, and then add it to the remaining number. Repeat this until a 10-digit number remains. Prove that this number has two equal digits.

12. Three integers are written on a blackboard. If a, b, and c are written, we can replace a with $b + c - 1$. This is repeated many times until we get $17, 1967, 1983$. Could the initial numbers be

    (a) $2, 2, 2$

    (b) $3, 3, 3$

13. There are several a's, b's, and c's written on a blackboard. We may replace two c's by one c, two a's by one b, two b's by one a, an a and a b by one c, an a and a c by one a, a b and a c by one c. Eventually, there will be one letter left. Prove that this letter doesn't depend on the order of erasure.

14. On every square of a 2015×2015 board is written either 1 or -1. For each row and column, we compute the product of all numbers in that row or column. Prove that the sum of these products cannot be 0.

15. Start with the set $\{3, 4, 12\}$. In each step you may choose two of the numbers a and b, and replace them with $\frac{3a-4b}{5}$ and $\frac{4a+3b}{5}$. Can you reach the set $\{4, 6, 12\}$?

16. 2015 squares of a 2016×2016 grid are infected. Every day, any square adjacent to two infected squares will also become infected. Show that at least one square stays uninfected.

17. All diagonals of a regular (a) pentagon (b) hexagon are drawn. Each vertex and each point of intersection of the diagonals is labeled by the number 1. In one step you can change the signs of all numbers of a side or diagonal. Is it possible to change the signs of all labels to -1 by a sequence of steps?

18. A lock has 16 keys arranged in a 4×4 array, each key oriented either horizontally or vertically. In order to open it, all the keys must be vertically oriented. When a key is switched to another position, all the other keys in the same row and column automatically switch their positions too. Show that the lock can always be opened, regardless of the starting position.

19. The integers $1, 2, \ldots, n$ are arranged in any order. In one step you may switch any two neighboring integers. Prove that you can never return to the initial order after an odd number of steps.

20. Write n 1's on a blackboard. You may erase any two of the numbers, say a and b, and write $\frac{a+b}{4}$ instead. After repeating this $n-1$ times, there is one number left. Prove that this last number is greater or equal to $\frac{1}{n}$

21. In a 4×4 grid, all entries are 1, except for the second one on the top row, which is -1. You may switch the signs of all numbers of a row, column, or a parallel to one of the diagonals. Prove that at least one -1 will remain in the table.

22. A deck has n cards labeled $1, 2, \ldots, n$. Starting with the deck in any order, repeat the following operation: if the card on top is labeled k, reverse the order of the first k cards. Prove that eventually the first card will be 1 (so no further changes occur).

23. There are n red points and n blue points in the plane, with no 3 points collinear. Show that we can draw n segments joining the blue points to the red points such that no pair of segments intersect.

24. Consider all points (x, y), where x and y are nonnegative integers. Shade the points $(0, 0)$, $(1, 0)$, $(0, 1)$, $(2, 0)$, $(1, 1)$, and $(0, 2)$.

    (a) Each of the shaded points have one chip on them.

    (b) Only the point $(0, 0)$ has a chip on it.

    You can repeat the following step: if there is a chip at (x, y), but $(x+1, y)$ and $(x, y+1)$ are free, you can remove the chip from (x, y) and place a chip on each of $(x+1, y)$ and $(x, y+1)$. In either starting position, can you remove the chips from the shaded points in a finite number of moves?

Hints

1. There are several invariants you could use. One of them is similar to Example 1.
2. Parity of the number of black squares.
3. Each term's parity is invariant.
4. Parity of the number of handshakes each person has had.
5. Use mod 9 (think of the divisibility rule for 9).
6. The knight alternates between white and black squares.
7. Sum of the four corners.
8. Each domino covers one black and one white square.
9. If there are x particles of type X, y of type Y, and z of type Z, how do the differences $x - y$, $y - z$, and $z - x$ change?
10. Sum of squares.
11. How does the number change in mod 9?
12. Parity of each term.
13. Difference between the number of a's and b's.
14. What happens to the sum when we change the sign of one square?
15. Consider the sum of their squares.
16. Show that the perimeter of the infected area is a monovariant.
17. If we find an odd number of points whose product is invariant, does this solve the problem?
18. Let x be the number of squares with an odd number of horizontal keys in its row or column (including the square itself). Can we decrease x until it is 0?
19. Look at the number of pairs (a, b) that are out of order (i.e. a comes before b, but $a > b$).
20. Hint 1: Consider the reciprocals of the terms.
    Hint 2: The sum of the reciprocals is a monovariant.
21. Hint 1: The product of certain squares is invariant.
    Hint 2: One of these squares is the original location of the -1.
22. What happens to the position of card n after it appears on the top?
23. Hint 1: Let points A and B be blue, and C and D be red. If AC intersects BD, we can replace them with AD and BC.
    Hint 2: Is there a monovariant that decreases as we do this, to prove that it will end?
24. Assign the number $\frac{1}{2^{x+y}}$ to point (x, y).